

# FROBENIUS THEORY FAILS FOR SEMIGROUPS OF POSITIVE MAPS ON VON NEUMANN ALGEBRAS

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ABSTRACT. The eigenvectors of an ergodic semigroup of linear normal positive unital maps on a von Neumann algebra are described. Moreover, it is shown by means of examples, that mere positivity of the maps in question is not sufficient for Frobenius theory as in [1] to hold.

## 1. INTRODUCTION

Frobenius theory for completely positive maps on von Neumann algebras was developed in [1] (further contributions to this subject can be found in [2] and [5]). This theory states, in particular, that for an ergodic semigroup of completely positive (or, in fact, even Schwarz) maps on a von Neumann algebra its point spectrum forms a group, and the corresponding eigenspaces are one-dimensional and spanned by a unitary operator. The aim of this paper is to show that neither of these is true for a semigroup of merely positive maps. Namely, we first prove that the eigenvectors are either multiples of a partial isometry or linear combinations of two partial isometries or multiples of a unitary operator, and then show by means of examples, that there is a whole class of semigroups of positive maps on a von Neumann algebra such that their point spectra are not groups and the corresponding eigenspaces have dimensions greater than one.

## 2. PRELIMINARIES AND NOTATION

Let  $M$  be a von Neumann algebra, and let  $(\Phi_g: g \in \mathbb{G})$  be a semigroup of linear normal positive unital maps on  $M$ . We shall be concerned with two cases:  $\mathbb{G} = \mathbb{N}_0$  — all nonnegative integers, and  $\mathbb{G} = \mathbb{R}_+$  — all nonnegative reals (notice that in the first case the semigroup has the form  $(\Phi^n: n = 0, 1, \dots)$ , where  $\Phi$  is a linear normal positive unital map on  $M$ ).

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1991 *Mathematics Subject Classification*. Primary: 46L55; Secondary: 28D05.

*Key words and phrases*. Frobenius theory, positive maps on von Neumann algebras, eigenvalues and eigenspaces.

Work supported by KBN grant 2PO3A 03024.

A complex number  $\lambda$  of modulus one is called an *eigenvalue* of the semigroup, if there is a nonzero  $x \in M$  such that for each  $g \in \mathbb{G}$

$$(1) \quad \Phi_g(x) = \lambda^g x.$$

The collection of all  $x$ 's such that (1) holds is called the *eigenspace* corresponding to the eigenvalue  $\lambda$ , and denoted by  $M_\lambda$ . In particular,  $M_1$  is the fixed-point space of the semigroup, and the semigroup is called *ergodic* if  $M_1$  consists of multiples of the identity. The set of all eigenvalues of the semigroup is called its *point spectrum*, and denoted by  $\sigma((\Phi_g))$ . Let  $\omega$  be a normal faithful state on  $M$  such that for each  $g \in \mathbb{G}$ ,  $\omega \circ \Phi_g = \omega$ . The part of Frobenius theory developed in [1] which is of interest to us, states that in this case, if we assume that  $(\Phi_g : g \in \mathbb{G})$  is ergodic and the maps  $\Phi_g$  are two-positive, the point spectrum is a group, and the corresponding eigenspaces are one-dimensional and spanned by a unitary operator. A natural question is if the same is true under the assumption of mere positivity of the maps  $\Phi_g$ . We shall show that this is not the case neither for the group structure nor for the dimension of the eigenspaces.

Let  $N$  be the  $\sigma$ -weak closure of the linear span of  $\bigcup_\lambda M_\lambda$ , where the sum is taken over all eigenvalues of  $(\Phi_g)$ . Then according to [3, Theorem 1]  $N$  is a  $JW^*$ -algebra, by which is meant that  $N$  is a  $\sigma$ -weakly closed linear space, closed with respect to the Jordan product

$$x \circ y = \frac{1}{2}(xy + yx);$$

moreover,  $\Phi_g|_N$  are Jordan  $*$ -automorphisms. Consequently, if  $x \in M_\lambda$ , then

$$\Phi_g(x^*) = \Phi_g(x)^* = \bar{\lambda}^g x^*,$$

meaning that  $x^* \in M_{\bar{\lambda}}$ , and for  $x \in M_{\lambda_1}$ ,  $y \in M_{\lambda_2}$  we have

$$\Phi_g(x \circ y) = \Phi_g(x) \circ \Phi_g(y) = \lambda_1^g x \circ \lambda_2^g y = (\lambda_1 \lambda_2)^g x \circ y,$$

meaning that  $x \circ y \in M_{\lambda_1 \lambda_2}$ . In particular, for an eigenvector  $x$  we have  $x \circ x^* \in M_1$ .

### 3. EIGENVECTORS

In the following theorem we shall describe the eigenvectors of  $(\Phi_g)$ .

**Theorem.** *Assume that  $(\Phi_g)$  is ergodic, and let  $x \in M_\lambda$  be an eigenvector of  $(\Phi_g)$ . Then one of the following possibilities holds:*

- (i)  $x = \alpha v$ , where  $\alpha \in \mathbb{C}$ , and  $v$  is a partial isometry in  $M_\lambda$  such that

$$v^* v = e, \quad v v^* = e^\perp$$

for some nonzero projection  $e$ ,  $e \neq \mathbf{1}$ ;

- (ii)  $x = \alpha_1 v_1 + \alpha_2 v_2$ , where  $\alpha_1, \alpha_2 \in \mathbb{C}$ ,  $\alpha_1 \neq \alpha_2$ , and  $v_1, v_2$  are partial isometries in  $M_\lambda$  such that for some nonzero projection  $e$ ,  $e \neq \mathbf{1}$ ,

$$v_1^* v_1 = e, \quad v_1 v_1^* = e^\perp, \quad v_2^* v_2 = e^\perp, \quad v_2 v_2^* = e;$$

- (iii)  $x = \alpha u$ , where  $\alpha \in \mathbb{C}$ , and  $u$  is a unitary operator in  $M_\lambda$ .

*Proof.* We have  $x \circ x^* \in M_1$ , so renorming  $x$ , if necessary, we may assume that

$$(2) \quad x^* x + x x^* = \mathbf{1}.$$

Multiplying both sides of the above equality by  $x$  on the left and on the right, respectively, we obtain

$$(3) \quad x x^* x + x^2 x^* = x = x^* x^2 + x x^* x,$$

so in particular  $x^*$  commutes with  $x^2$ . Multiplying again both sides of the second part of the above equality by  $x^*$ , we obtain

$$(4) \quad x^* x x^* x + x^{*2} x^2 = x^* x.$$

- (i) Assume that  $x^2 = 0$ . Then (4) gives

$$(x^* x)^2 = x^* x,$$

so  $x^* x$  is a projection  $e$ , and by (2)  $x x^* = e^\perp$ , thus  $x$  is a partial isometry with initial projection  $e$  and final projection  $e^\perp$ , hence part (i) of the conclusion of the theorem follows.

(ii) and (iii) Assume that  $x^2 \neq 0$ . Then  $x^2 \in M_{\lambda^2}$ ,  $x^{*2} \in M_{\bar{\lambda}^2}$ , so  $x^2 \circ x^{*2} \in M_1$ , and since  $x^2$  and  $x^{*2}$  commute, we get

$$x^2 x^{*2} = x^2 \circ x^{*2} = \theta \mathbf{1}$$

for some  $\theta > 0$ , by the assumed ergodicity of  $(\Phi_g)$ . Denote  $z = x^* x$ . Then the above equality and (4) give

$$z^2 - z + \theta \mathbf{1} = 0,$$

that is

$$\left( z - \frac{1 - \sqrt{1 - 4\theta}}{2} \mathbf{1} \right) \left( z - \frac{1 + \sqrt{1 - 4\theta}}{2} \mathbf{1} \right) = 0.$$

It follows that  $\theta \leq 1/4$ , and we have two possibilities for the spectrum of  $z$ : either  $\theta = 1/4$  in which case  $\text{sp } z = \{1/2\}$ , or  $\theta < 1/4$  in which case  $\text{sp } z = \left\{ \frac{1 - \sqrt{1 - 4\theta}}{2}, \frac{1 + \sqrt{1 - 4\theta}}{2} \right\}$ . The first possibility gives at once

$$x^* x = \frac{1}{2} \mathbf{1}, \quad x x^* = \frac{1}{2} \mathbf{1},$$

and part (iii) of the conclusion of the theorem follows.

For the second possibility we have

$$x^*x = \frac{1 - \sqrt{1 - 4\theta}}{2}e + \frac{1 + \sqrt{1 - 4\theta}}{2}e^\perp$$

where  $e$  and  $e^\perp$  are the spectral projections of  $x^*x$ . From (2) we get

$$xx^* = \frac{1 + \sqrt{1 - 4\theta}}{2}e + \frac{1 - \sqrt{1 - 4\theta}}{2}e^\perp,$$

and denoting

$$\alpha_1 = \left( \frac{1 - \sqrt{1 - 4\theta}}{2} \right)^{1/2}, \quad \alpha_2 = \left( \frac{1 + \sqrt{1 - 4\theta}}{2} \right)^{1/2},$$

we obtain

$$(5) \quad \begin{aligned} |x| &= \alpha_1 e + \alpha_2 e^\perp \\ |x^*| &= \alpha_2 e + \alpha_1 e^\perp. \end{aligned}$$

Let

$$x = u|x|$$

be the polar decomposition of  $x$ . Then

$$x^* = u^*|x^*|$$

is the polar decomposition of  $x^*$ , and since  $|x|$  and  $|x^*|$  are invertible the operator  $u$  is unitary. Moreover,

$$|x^*| = u|x|u^*,$$

which gives the equality

$$\alpha_2 e + \alpha_1 e^\perp = \alpha_1 u e u^* + \alpha_2 u e^\perp u^*,$$

yielding

$$\alpha_1 \mathbf{1} + (\alpha_2 - \alpha_1)e = \alpha_2 \mathbf{1} + (\alpha_1 - \alpha_2)u e u^*.$$

Thus

$$(\alpha_1 - \alpha_2)\mathbf{1} = (\alpha_1 - \alpha_2)(e + u e u^*),$$

showing that

$$u e u^* = e^\perp,$$

and consequently,

$$e = \mathbf{1} - u e u^* = u e^\perp u^*.$$

From the polar decomposition and formula (5) we obtain

$$(6) \quad x = \alpha_1 u e + \alpha_2 u e^\perp = \alpha_1 v_1 + \alpha_2 v_2,$$

where

$$v_1 = u e, \quad v_2 = u e^\perp.$$

We have

$$v_1^* v_1 = e, \quad v_1 v_1^* = u e u^* = e^\perp, \quad v_2^* v_2 = e^\perp, \quad v_2 v_2^* = u e^\perp u^* = e,$$

so (6) is the representation of  $x$  as in part (ii) of the conclusion of the theorem. It remains to show that  $v_1, v_2 \in M_\lambda$ . Equality (3) gives

$$x x^* x = x - x^2 x^* = x - x^2 \circ x^*$$

because  $x^2$  and  $x^*$  commute. Since  $x^2 \in M_{\lambda^2}$  and  $x^* \in M_{\bar{\lambda}}$ , we get  $x^2 \circ x^* \in M_\lambda$ , so  $x x^* x \in M_\lambda$ . We have

$$x x^* x = (\alpha_1 v_1 + \alpha_2 v_2)(\alpha_1^2 e + \alpha_2^2 e^\perp) = \alpha_1^3 v_1 + \alpha_2^3 v_2,$$

and the equality

$$\Phi_g(x x^* x) = \lambda^g x x^* x$$

yields

$$(7) \quad \alpha_1^3 \Phi_g(v_1) + \alpha_2^3 \Phi_g(v_2) = \lambda^g \alpha_1^3 v_1 + \lambda^g \alpha_2^3 v_2.$$

On the other hand  $\Phi_g(x) = \lambda^g x$ , which gives

$$(8) \quad \alpha_1 \Phi_g(v_1) + \alpha_2 \Phi_g(v_2) = \lambda^g \alpha_1 v_1 + \lambda^g \alpha_2 v_2.$$

Multiplying both sides of equality (8) by  $\alpha_2^2$  and subtracting (8) from (7) we obtain

$$\alpha_1(\alpha_1^2 - \alpha_2^2) \Phi_g(v_1) = \lambda^g \alpha_1(\alpha_1^2 - \alpha_2^2) v_1,$$

which gives  $\Phi_g(v_1) = \lambda^g v_1$ , and analogously  $\Phi_g(v_2) = \lambda^g v_2$ .  $\square$

#### 4. POSITIVITY OF ELEMENTS FROM $\mathbf{Mat}_2(M)$

In what follows we shall need a number of properties concerning positivity of matrices with elements in a von Neumann algebra. They will be exploited in examples in a particular case of an abelian von Neumann algebra, but as these properties seem to be interesting in their own right, we prove them here in slightly greater generality. For more information on positivity of such matrices the reader is referred to [4, Chapter IV.3] and [6]. It should be added that some of the facts obtained below can be given alternative proofs based on methods used in [4, 6].

Let  $M$  be a von Neumann algebra, and let  $\widetilde{M} = \mathbf{Mat}_2(M)$  be the algebra of  $2 \times 2$ -matrices with elements from  $M$ . Assuming that  $M$  acts on a Hilbert space  $\mathcal{H}$ , we can consider  $\widetilde{M}$  as acting on the Hilbert space  $\widetilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$ .

**Proposition 1.** *Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \widetilde{M}$ .*

- (i)  *$A \geq 0$  if and only if  $a, d \geq 0$ ,  $c = b^*$ , and for each  $\varepsilon > 0$   $d \geq b^*(a + \varepsilon \mathbf{1})^{-1} b$ .*

- (ii) Assume that  $a, b$  and  $c$  commute. Then  $A \geq 0$  if and only if  $a, d \geq 0$ ,  $c = b^*$ , and  $ad \geq b^*b$ .

*Proof.* Calculate first the quadratic form of  $A$ . For  $\tilde{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ ,  $\xi_1, \xi_2 \in \mathcal{H}$ , we have

$$(9) \quad \begin{aligned} \langle A\tilde{\xi}, \tilde{\xi} \rangle_{\tilde{\mathcal{H}}} &= \left\langle \begin{pmatrix} a\xi_1 + b\xi_2 \\ c\xi_1 + d\xi_2 \end{pmatrix}, \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \right\rangle_{\tilde{\mathcal{H}}} \\ &= \langle a\xi_1, \xi_1 \rangle_{\mathcal{H}} + \langle b\xi_2, \xi_1 \rangle_{\mathcal{H}} + \langle c\xi_1, \xi_2 \rangle_{\mathcal{H}} + \langle d\xi_2, \xi_2 \rangle_{\mathcal{H}}. \end{aligned}$$

It is clear that in order that  $A$  be positive we must have  $a, d \geq 0$  and  $c = b^*$ , so we restrict attention to matrices  $A$  of the form  $A = \begin{bmatrix} a & b \\ b^* & d \end{bmatrix}$  with  $a, d \geq 0$ . Then (9) becomes

$$(10) \quad \langle A\tilde{\xi}, \tilde{\xi} \rangle_{\tilde{\mathcal{H}}} = \langle a\xi_1, \xi_1 \rangle_{\mathcal{H}} + \langle b\xi_2, \xi_1 \rangle_{\mathcal{H}} + \langle \xi_1, b\xi_2 \rangle_{\mathcal{H}} + \langle d\xi_2, \xi_2 \rangle_{\mathcal{H}}.$$

(i) *Step 1.* First we shall show that for  $A$  of the form  $A = \begin{bmatrix} 1 & b \\ b^* & z \end{bmatrix}$ ,  $A \geq 0$  if and only if  $z \geq b^*b$ . This is virtually proved in [6, Lemma 3.1]. For the sake of completeness we give a simple proof below.

For the quadratic form of  $A$  we have

$$(11) \quad \langle A\tilde{\xi}, \tilde{\xi} \rangle_{\tilde{\mathcal{H}}} = \langle \xi_1, \xi_1 \rangle_{\mathcal{H}} + \langle b\xi_2, \xi_1 \rangle_{\mathcal{H}} + \langle \xi_1, b\xi_2 \rangle_{\mathcal{H}} + \langle z\xi_2, \xi_2 \rangle_{\mathcal{H}}.$$

Let  $b^*b \leq z$ . Then

$$\begin{aligned} \langle A\tilde{\xi}, \tilde{\xi} \rangle_{\tilde{\mathcal{H}}} &\geq \langle \xi_1, \xi_1 \rangle_{\mathcal{H}} + \langle b\xi_2, \xi_1 \rangle_{\mathcal{H}} + \langle \xi_1, b\xi_2 \rangle_{\mathcal{H}} + \langle b\xi_2, b\xi_2 \rangle_{\mathcal{H}} \\ &= \langle \xi_1 + b\xi_2, \xi_1 + b\xi_2 \rangle_{\mathcal{H}} \geq 0, \end{aligned}$$

showing that  $A \geq 0$ .

Conversely, if  $A \geq 0$  then substituting  $-\xi_1$  for  $\xi_1$  in (11), we obtain

$$0 \leq \langle \xi_1, \xi_1 \rangle_{\mathcal{H}} - \langle b\xi_2, \xi_1 \rangle_{\mathcal{H}} - \langle \xi_1, b\xi_2 \rangle_{\mathcal{H}} + \langle z\xi_2, \xi_2 \rangle_{\mathcal{H}},$$

that is

$$\langle b\xi_2, \xi_1 \rangle_{\mathcal{H}} + \langle \xi_1, b\xi_2 \rangle_{\mathcal{H}} - \langle \xi_1, \xi_1 \rangle_{\mathcal{H}} \leq \langle z\xi_2, \xi_2 \rangle_{\mathcal{H}}.$$

Now putting  $\xi_1 = b\xi_2$  in the above inequality, we get

$$\langle b\xi_2, b\xi_2 \rangle_{\mathcal{H}} \leq \langle z\xi_2, \xi_2 \rangle_{\mathcal{H}},$$

which shows that  $b^*b \leq z$ .

*Step 2.* Let now  $A = \begin{bmatrix} a & b \\ b^* & d \end{bmatrix}$  be arbitrary.  $A \geq 0$  if and only if for each  $\varepsilon > 0$ ,  $A_\varepsilon = \begin{bmatrix} a+\varepsilon\mathbf{1} & b \\ b^* & d \end{bmatrix} \geq 0$ , and denoting  $a_\varepsilon = a + \varepsilon\mathbf{1}$ , we obtain

$$\langle A_\varepsilon \tilde{\xi}, \tilde{\xi} \rangle_{\tilde{\mathcal{H}}} = \langle a_\varepsilon \xi_1, \xi_1 \rangle_{\mathcal{H}} + \langle b\xi_2, \xi_1 \rangle_{\mathcal{H}} + \langle \xi_1, b\xi_2 \rangle_{\mathcal{H}} + \langle d\xi_2, \xi_2 \rangle_{\mathcal{H}}.$$

Putting  $\eta_1 = a_\varepsilon^{1/2} \xi_1$ , we get

$$\begin{aligned} \langle A_\varepsilon \tilde{\xi}, \tilde{\xi} \rangle_{\tilde{\mathcal{H}}} &= \langle \eta_1, \eta_1 \rangle_{\mathcal{H}} + \langle a_\varepsilon^{-1/2} b \xi_2, \eta_1 \rangle_{\mathcal{H}} + \langle \eta_1, a_\varepsilon^{-1/2} b \xi_2 \rangle_{\mathcal{H}} + \langle d \xi_2, \xi_2 \rangle_{\mathcal{H}} \\ &= \left\langle \begin{bmatrix} \mathbf{1} & a_\varepsilon^{-1/2} b \\ b^* a_\varepsilon^{-1/2} & d \end{bmatrix} \begin{pmatrix} \eta_1 \\ \xi_2 \end{pmatrix}, \begin{pmatrix} \eta_1 \\ \xi_2 \end{pmatrix} \right\rangle_{\tilde{\mathcal{H}}}. \end{aligned}$$

Since  $a_\varepsilon^{1/2}$  maps  $\mathcal{H}$  onto  $\tilde{\mathcal{H}}$  in a 1-1 way, we see that  $A_\varepsilon \geq 0$  if and only if  $\begin{bmatrix} \mathbf{1} & a_\varepsilon^{-1/2} b \\ b^* a_\varepsilon^{-1/2} & d \end{bmatrix} \geq 0$ , which by *Step 1* is equivalent to the condition

$$d \geq b^* a_\varepsilon^{-1} b = b^* (a + \varepsilon \mathbf{1})^{-1} b,$$

and the proof of (i) is complete.

(ii) The reasoning is similar to that in part (i) using the simple  $\varepsilon$ -trick. Namely, the inequality  $ad \geq b^* b$  is equivalent to  $(a + \varepsilon \mathbf{1})d \geq b^* b$  for each  $\varepsilon > 0$ , which in turn, by the assumed commutation property, is equivalent to  $d \geq b^* (a + \varepsilon \mathbf{1})^{-1} b$ . Applying part (i) finishes the proof.  $\square$

**Remark.** In virtually the same way we obtain the following variant of (i):

(i')  $A \geq 0$  if and only if  $a, d \geq 0$ ,  $c = b^*$ , and for each  $\varepsilon > 0$   $a \geq b(d + \varepsilon \mathbf{1})^{-1} b^*$ .

**Lemma 2.** Let  $\begin{bmatrix} a & b \\ b^* & d \end{bmatrix} \geq 0$ . Then

- (i)  $\begin{bmatrix} d & b^* \\ b & a \end{bmatrix} \geq 0$ .
- (ii) For each  $x, y \in M$   $\begin{bmatrix} xax^* & xby^* \\ yb^*x^* & ydy^* \end{bmatrix} \geq 0$ .

*Proof.* (i) follows from the equality

$$\begin{bmatrix} d & b^* \\ b & a \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{bmatrix} \begin{bmatrix} a & b \\ b^* & d \end{bmatrix} \begin{bmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{bmatrix},$$

and (ii) from the equality

$$\begin{bmatrix} xax^* & xby^* \\ yb^*x^* & ydy^* \end{bmatrix} = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \begin{bmatrix} a & b \\ b^* & d \end{bmatrix} \begin{bmatrix} x^* & 0 \\ 0 & y^* \end{bmatrix}.$$

$\square$

**Lemma 3.** Let  $a$  commute with  $b$ , and assume further that either  $b$  is normal or that  $b$  commutes with  $d$ . If  $\begin{bmatrix} a & b \\ b^* & d \end{bmatrix} \geq 0$ , then  $\begin{bmatrix} a & b^* \\ b & d \end{bmatrix} \geq 0$ .

*Proof.* Let  $\begin{bmatrix} a & b \\ b^* & d \end{bmatrix} \geq 0$ , and assume first that  $b$  is normal. Then by Proposition 1 (i) we have for each  $\varepsilon > 0$

$$d \geq b^* (a + \varepsilon \mathbf{1})^{-1} b = b^* b (a + \varepsilon \mathbf{1})^{-1} = b (a + \varepsilon \mathbf{1})^{-1} b^*,$$

which again by Proposition 1 (i) means that  $\begin{bmatrix} a & b^* \\ b & d \end{bmatrix} \geq 0$ .

Now let  $b$  commute with  $d$ . Then for each  $\varepsilon > 0$ ,

$$d \geq b^*(a + \varepsilon \mathbf{1})^{-1}b = |b|^2(a + \varepsilon \mathbf{1})^{-1} = |b|(a + \varepsilon \mathbf{1})^{-1}|b|,$$

so  $\begin{bmatrix} a & |b| \\ |b| & d \end{bmatrix} \geq 0$ . By Lemma 2 (i) it follows that  $\begin{bmatrix} d & |b| \\ |b| & a \end{bmatrix} \geq 0$ , and thus, on account of Proposition 1 (i), for each  $\varepsilon > 0$ ,

$$a \geq |b|(d + \varepsilon \mathbf{1})^{-1}|b| = b^*b(d + \varepsilon \mathbf{1})^{-1} = b^*(d + \varepsilon \mathbf{1})^{-1}b,$$

which means that  $\begin{bmatrix} d & b \\ b^* & a \end{bmatrix} \geq 0$ . Applying again Lemma 2 (i) we obtain  $\begin{bmatrix} a & b^* \\ b & d \end{bmatrix} \geq 0$ .  $\square$

## 5. EXAMPLES

Let us begin with a simple example.

**Example 1.** Keeping the notation of Section 4, put  $\widetilde{M} = \mathbb{B}(\mathbb{C}^2)$  (i.e.  $M = \mathbb{C}$ ),  $\omega = \frac{1}{2}tr$ , and let  $\lambda_0 \in \mathbb{C}$  be such that  $|\lambda_0| = 1$ ,  $\lambda_0 \neq 1$ . Define  $\Phi : \widetilde{M} \rightarrow \widetilde{M}$  as

$$\Phi \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} \frac{a+d}{2} & \lambda_0 b \\ \bar{\lambda}_0 c & \frac{a+d}{2} \end{bmatrix}.$$

It is clear that  $\Phi$  is linear normal unital, and that  $\omega \circ \Phi = \omega$ . Moreover, for  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \geq 0$ , we have  $a, d \geq 0$ ,  $c = \bar{b}$ , and  $\begin{bmatrix} d & b \\ \bar{b} & a \end{bmatrix} \geq 0$ , thus  $\begin{bmatrix} \frac{a+d}{2} & 2b \\ 2\bar{b} & a+d \end{bmatrix} \geq 0$ , so  $\begin{bmatrix} \frac{a+d}{2} & b \\ \bar{b} & \frac{a+d}{2} \end{bmatrix} \geq 0$ , and consequently  $\begin{bmatrix} \frac{a+d}{2} & \lambda_0 b \\ \bar{\lambda}_0 \bar{b} & \frac{a+d}{2} \end{bmatrix} \geq 0$ , showing that  $\Phi$  is positive.

The equality

$$\Phi \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

yields

$$\frac{a+d}{2} = a = d, \quad \lambda_0 b = b, \quad \bar{\lambda}_0 c = c,$$

hence  $b = c = 0$ , and the fixed-points have the form  $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ , which means that  $(\Phi^n)$  is ergodic.

(i) Let  $\lambda_0$  be such that  $\lambda_0 \neq -1$ ,  $\lambda_0^3 \neq 1$ , and let  $\lambda \neq 1$  be an eigenvalue of  $(\Phi^n)$ . Then

$$\frac{a+d}{2} = \lambda a = \lambda d, \quad \lambda_0 b = \lambda b, \quad \bar{\lambda}_0 c = \lambda c,$$

which yields

$$a = d = 0, \quad \bar{\lambda} \lambda_0 b = b, \quad \bar{\lambda} \bar{\lambda}_0 c = c.$$



Thus either  $\lambda = \lambda_0$ ,  $c = 0$  or  $\lambda = \bar{\lambda}_0$ ,  $b = 0$ , so  $\lambda_0$  and  $\bar{\lambda}_0$  are the only eigenvalues of  $(\Phi^n)$  different from 1, with the eigenspaces

$$\widetilde{M}_{\lambda_0} = \left\{ \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} : b \in \mathbb{C} \right\}, \quad \widetilde{M}_{\bar{\lambda}_0} = \left\{ \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} : c \in \mathbb{C} \right\}.$$

Consequently,  $\sigma((\Phi^n)) = \{1, \lambda_0, \bar{\lambda}_0\}$ , which is not a group if  $\lambda_0 \neq -1$ ,  $\lambda_0^3 \neq 1$ .

(ii) Now let  $\lambda_0 = -1$ . The above calculations give  $\sigma((\Phi^n)) = \{1, -1\}$ , and

$$\widetilde{M}_{-1} = \left\{ \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} : b, c \in \mathbb{C} \right\},$$

so the eigenspace is not one-dimensional.  $\square$

Let us observe that the reasoning above may be repeated with virtually no change for the semigroup  $(\Phi_t : t \geq 0)$  defined as

$$\Phi_t \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} \frac{a+d}{2} & \lambda_0^t b \\ \bar{\lambda}_0^t c & \frac{a+d}{2} \end{bmatrix},$$

thus giving a corresponding example in the continuous case.

**Remark.** The triple  $(\mathbb{B}(\mathbb{C}^2), (\Phi_t), \omega)$  from the above example constitutes what in [2] is called an irreducible  $W^*$ -dynamical system. In [2, Theorem 3.8] it is proved that under the assumption that the  $\Phi_t$ 's are Schwarz maps every such a system on a full algebra has trivial point spectrum (i.e. consisting only of 1). As we see this is not the case if we assume only positivity of the maps  $\Phi_t$ 's.

Now we construct a more involved example (in fact, a class of examples) in which we shall see that all the possibilities for the eigenvectors given in Theorem may occur.

**Example 2.** Let  $M$  be abelian, let  $\omega$  be a normal faithful state on  $M$ , and let  $\Psi$  be a positive normal unital map on  $M$  such that  $\omega \circ \Psi = \omega$ ,  $(\Psi^n)$  is ergodic, and  $\sigma((\Psi^n)) = \{-1, 1\}$ . The abelianness of  $M$  implies that  $\Psi$  is completely positive (cf. [4, Chapter IV.3]), thus according to [1] the eigenspace corresponding to  $-1$  is one-dimensional and spanned by a unitary operator  $u$ , i.e.

$$\Psi(x) = -x$$

if and only if  $x$  is a multiple of  $u$ .

Put  $\widetilde{M} = \mathbf{Mat}_2(M)$ ,

$$\tilde{\omega} \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \frac{1}{2}[\omega(a) + \omega(d)],$$

and let  $\lambda_0 \in \mathbb{C}$  be such that  $|\lambda_0| = 1$ ,  $\lambda_0 \notin \{-1, 1\}$ . Define  $\Phi : \widetilde{M} \rightarrow \widetilde{M}$  as

$$\Phi \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} \Psi \left( \frac{a+d}{2} \right) & \lambda_0 \Psi(b) \\ \bar{\lambda}_0 \Psi(c) & \Psi \left( \frac{a+d}{2} \right) \end{bmatrix}.$$

$\Phi$  is a linear normal unital map on  $\widetilde{M}$  such that  $\tilde{\omega} \circ \Phi = \tilde{\omega}$ . Arguing as in Example 1, and using Lemma 2 with  $x = \lambda_0 \mathbf{1}$ ,  $y = \mathbf{1}$ , and Lemma 3, we see that if  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \geq 0$ , then  $\begin{bmatrix} \frac{a+d}{2} & \lambda_0 b \\ \bar{\lambda}_0 c & \frac{a+d}{2} \end{bmatrix} \geq 0$ , so  $\Phi$  is positive by virtue of the complete positivity of  $\Psi$ .

The equality

$$\Phi \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

yields

$$\Psi \left( \frac{a+d}{2} \right) = a = d, \quad \lambda_0 \Psi(b) = b, \quad \bar{\lambda}_0 \Psi(c) = c,$$

hence  $b = c = 0$  and  $\Psi(a) = a$ . From the ergodicity of  $\Psi$  it follows that  $a$  is a multiple of  $\mathbf{1}$ , so the fixed-points of  $(\Phi^n)$  have the form  $\begin{bmatrix} \theta \mathbf{1} & 0 \\ 0 & \theta \mathbf{1} \end{bmatrix}$  with  $\theta \in \mathbb{C}$ , which means that  $(\Phi^n)$  is ergodic. Let  $\lambda \neq 1$  be an eigenvalue of  $(\Phi^n)$ . Then

$$\Psi \left( \frac{a+d}{2} \right) = \lambda a = \lambda d, \quad \lambda_0 \Psi(b) = \lambda b, \quad \bar{\lambda}_0 \Psi(c) = \lambda c,$$

which yields

$$(12) \quad \Psi(a) = \lambda a, \quad \Psi(b) = \bar{\lambda}_0 \lambda b, \quad \Psi(c) = \bar{\lambda}_0 \lambda c.$$

(i) Take  $\lambda_0$  such that  $\lambda_0 \neq i$ ,  $\lambda_0 \neq -i$ ,  $\lambda_0^3 \neq 1$ ,  $\lambda_0^3 \neq -1$ . Equalities (12) yield the following possibilities:

(i.1)  $\lambda = -1$ . Then  $a = d$  is a multiple of  $u$ ,  $b = c = 0$ , and

$$\widetilde{M}_{-1} = \left\{ \alpha \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} : \alpha \in \mathbb{C} \right\},$$

so the eigenvector corresponding to the eigenvalue  $-1$  is as in part (iii) of Theorem.

(i.2)  $\lambda \neq -1$ . Then  $a = d = 0$ , and one of the four situations must occur:

(i.2.1)  $\lambda = \lambda_0$ . Then  $b$  is a multiple of  $\mathbf{1}$ ,  $c = 0$ , and

$$\widetilde{M}_{\lambda_0} = \left\{ \alpha \begin{bmatrix} 0 & \mathbf{1} \\ 0 & 0 \end{bmatrix} : \alpha \in \mathbb{C} \right\},$$

so the eigenvector corresponding to the eigenvalue  $\lambda_0$  is as in part (i) of Theorem.

(i.2.2)  $\lambda = -\lambda_0$ . Then  $b$  is a multiple of  $u$ ,  $c = 0$ , and

$$\widetilde{M}_{-\lambda_0} = \left\{ \alpha \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix} : \alpha \in \mathbb{C} \right\}.$$

(i.2.3)  $\lambda = \bar{\lambda}_0$ . Then  $b = 0$ ,  $c$  is a multiple of  $\mathbf{1}$ , and

$$\widetilde{M}_{\bar{\lambda}_0} = \left\{ \alpha \begin{bmatrix} 0 & 0 \\ \mathbf{1} & 0 \end{bmatrix} : \alpha \in \mathbb{C} \right\}.$$

(i.2.4)  $\lambda = -\bar{\lambda}_0$ . Then  $b = 0$ ,  $c$  is a multiple of  $u$ , and

$$\widetilde{M}_{-\bar{\lambda}_0} = \left\{ \alpha \begin{bmatrix} 0 & 0 \\ u & 0 \end{bmatrix} : \alpha \in \mathbb{C} \right\}.$$

Moreover, we have

$$\sigma((\Phi^n)) = \{1, -1, \lambda_0, \bar{\lambda}_0, -\lambda_0, -\bar{\lambda}_0\},$$

which is not a group under our assumptions on  $\lambda_0$ .

Now take  $\lambda_0 = i$ . Equations (12) become then

$$\Psi(a) = \lambda a, \quad \Psi(b) = -i\lambda b, \quad \Psi(c) = i\lambda c.$$

As in part (i) we have the possibilities:

(ii.1)  $\lambda = -1$ , in which case  $a = d$  is a multiple of  $\mathbf{1}$ , and  $b = c = 0$ .

(ii.2)  $\lambda \neq -1$ , in which case  $a = d = 0$ , and we may only have either  $\lambda = i$  or  $\lambda = -i$ . In the first case  $b$  is a multiple of  $\mathbf{1}$ ,  $c$  is a multiple of  $u$ , and

$$\widetilde{M}_i = \left\{ \alpha_1 \begin{bmatrix} 0 & \mathbf{1} \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 0 \\ u & 0 \end{bmatrix} : \alpha_1, \alpha_2 \in \mathbb{C} \right\},$$

so the situation is as in part (ii) of Theorem with

$$v_1 = \begin{bmatrix} 0 & \mathbf{1} \\ 0 & 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 & 0 \\ u & 0 \end{bmatrix}, \quad e = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{bmatrix}, \quad e^\perp = \begin{bmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{bmatrix}.$$

In the second case  $b$  is a multiple of  $u$ ,  $c$  is a multiple of  $\mathbf{1}$ , and

$$\widetilde{M}_{-i} = \left\{ \alpha_1 \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 0 \\ \mathbf{1} & 0 \end{bmatrix} : \alpha_1, \alpha_2 \in \mathbb{C} \right\},$$

so again part (ii) of Theorem occurs with

$$v_1 = \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 & 0 \\ \mathbf{1} & 0 \end{bmatrix}, \quad e = \begin{bmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{bmatrix}, \quad e^\perp = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{bmatrix}.$$

□

As in Example 1, we observe that taking  $(\Psi_t: t \geq 0)$  — an ergodic semigroup of positive maps on  $M$  with  $\sigma((\Psi_t)) = \{1, -1\}$ , and defining

$$\Phi_t \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} \Psi_t \left( \frac{a+d}{2} \right) & \lambda_0^t \Psi_t(b) \\ \bar{\lambda}_0^t \Psi_t(c) & \Psi_t \left( \frac{a+d}{2} \right) \end{bmatrix}, \quad t \geq 0,$$

we obtain a continuous counterpart of Example 2.

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